International Journal of Theoretical Physics, Vol. 38, No. 1, 1999

State Property Systems and Closure Spaces: A Study of Categorical Equivalence

Diederik Aerts,[1] Eva Colebunders,[1] Ann Van der Voorde,[1] and Bart Van Steirteghem[1]

Received August 7, 1998

We show that the natural mathematical structure to describe a physical entity by means of its states and its properties within the Geneva–Brussels approach is that of a state property system. We prove that the category of state property systems (and morphisms) **SP** is equivalent to the category of closure spaces (and continuous maps) **Cls**. We show the equivalence of the 'state determination axiom' for state property systems with the 'T_0 separation axiom' for closure spaces. We also prove that the category $\mathbf{SP_0}$ of state-determined state property systems is equivalent to the category $\mathbf{L_0}$ of based complete lattices. In this sense the equivalence of **SP** and **Cls** generalizes the equivalence of $\mathbf{Cls_0}$ (T_0 closure spaces) and $\mathbf{L_0}$ proven by Erné (1984).

1. INTRODUCTION

Constantin Piron started the elaboration of a realistic axiomatic theory for the foundations of quantum mechanics in Geneva and the first presentation of this approach can be found in Piron (1976). Apart from an axiomatic scheme presented in Piron (1976)—founded on his celebrated representation theorem (1964)—a first step of 'operational' foundation was exposed in Piron (1976) by introducing the concept of 'test' for a property. One of the authors of the present paper (D. Aerts) studied the problem of the description of 'separated physical entities' within this approach. Making use extensively of the 'operational' idea presented in Piron (1976), Aerts elaborated the 'operational' aspects of the theory, introducing a powerful 'calculus of tests' (Aerts, 1981, 1982). In this way the theory grew to a complete realistic and

[1] FUND and TOPO, Department of Mathematics, Brussels Free University, B-1050 Brussels, Belgium; e-mail: diraerts@vub.ac.be, evacoleb@vub.ac.be, avdvoord@vub.ac.be, bvsteirt@vub.ac.be.

operational theory, and the 'operational' part was very well suited to tackle 'physical' problems, like the description of separated entities (Aerts, 1981, 1982) and the filtering out of the classical part of an entity (Aerts, 1983). By now the theory has been further elaborated in Geneva and in Brussels and therefore we shall refer to it as the Geneva–Brussels approach to the foundations of physics. It is a 'realistic' and 'operational' theory, where a physical entity is described by means of its states and properties, and the properties are 'operationally' introduced as 'testable properties.'

The foundational material of the approach can be found in Piron (1976, 1989, 1990) and Aerts (1981, 1982, 1983) and we will therefore refer to these writings as the foundation papers. Meanwhile different problems have been investigated within the approach and connections with other approaches to the foundations of physics have been studied (Aerts, 1981, 1982, 1983, 1984, 1985, 1994, 1998; Aerts *et al.*, 1997; Aerts and Valckenborgh, 1998; Cattaneo *et al.*, 1988; Cattaneo and Nistico, 1991, 1992, 1993; Daniel, 1982; d'Emma, 1980; Foulis *et al.*, 1983; Foulis and Randall, 1984; Giovannini and Piron, 1979; Gisin, 1981; Jauch and Piron, 1965; Ludwig and Neumann, 1981; Moore, 1995; Piron, 1964, 1969, 1976, 1989, 1990; Randall and Foulis, 1983).

Although the foundational setting for the Geneva–Brussels approach was elaborated in Piron (1976, 1989, 1990) and Aerts (1981, 1982, 1983), the basic mathematical structure of the approach independent of the physical content had not yet been properly identified. This was started in Aerts (1998) within a more general setting and the resulting mathematical structure was called there a 'state property system' (Aerts, 1998; also see Section 2 of the present paper). It was shown—as we will do again in Section 2 of this paper—that the mathematical structure of a state property system, once the objects and morphisms are given their physical meaning, manages to represent all the subtleties of the approach. This has the enormous advantage that theorems can now be proven within the approach without using the 'physical interpretation, an indispensable step for a real formalization of the theory.

Moreover, it is proven that state property systems and their morphisms are in natural correspondence with closure spaces and continuous maps (Aerts, 1998). In the present paper we want to investigate this correspondence in detail: we show that the category of state property systems and its morphisms, which we call **SP**, is equivalent to the category **Cls** of closure spaces and continuous maps. This gives us a "lattice representation" for *all* closure spaces. It generalizes older (well-known) lattice representations where the closure spaces were (at least) T_0 (Erné, 1984): if we restrict ourselves to T_0 closure spaces, we recover the categorical equivalence between 'based complete lattices' and T_0 closure spaces, given in Erné (1984; see Sections 6 and 7 of the present paper).

The mathematical structure of a state property system that we will present in this paper appears as the formalization of a state property entity within the Geneva–Brussels approach. We remark, however, that it appears also as a fundamental mathematical structure in other situations where states and properties of physical entities are formalized [e.g., the situation presented in Aerts (1998) of an experiment state outcome entity with one experiment].

We remark that the description of a physical entity by means of its states and properties that we use in this article differs from the one in the founding papers (Piron, 1976, 1989, 1990; Aerts 1981, 1982, 1983) in two aspects:

1. We make an explicit distinction between the properties and the states. In the founding papers a state of an entity is represented by the set of all actual properties, making it impossible to introduce the distinction as we will do here. The distinction between states and properties was introduced in Aerts (1994), where it was shown that a condition of 'state determination' for an entity reduces this more general situation to the earlier one. It was also shown that the 'state determination' condition is equivalent to the T_0 separation axiom of the corresponding closure space. In Aerts (1994) the categorical equivalence between the description of an entity by means of states and properties and the representation in the corresponding closure space was not yet elaborated: this will be the main subject of the present paper.

2. We explicitly distinguish between the physical content and the mathematical form of the theory. This was not done systematically in the founding papers nor in Aerts (1994). In Aerts (1998), where such a systematic distinction between the physical and the mathematical is introduced for a more general theory also containing experiments and outcomes, the fruitfulness of this distinction became clear. It leads to the definition of the 'mathematical' concept of a 'state property system,' representing the states and the properties of a general physical entity. This concept will be the central mathematical "object" in the present paper. We will show in a forthcoming paper how the categories formulated in the present paper are connected to the categories presented in Moore (1995).

2. THE DESCRIPTION OF AN ENTITY BY MEANS OF ITS STATES AND TESTABLE PROPERTIES

Let us consider an entity S. The entity S is at every moment in a definite state p, and let us call Σ the well-defined set of considered states of the entity S.

If we have in mind a certain property a that the entity might have and if this property is testable, we can construct a test α for a. Such a test, also sometimes called 'question' or 'experimental project' in Piron (1976, 1989, 1990) and Aerts (1981, 1982, 1983), consists in an experiment that can be performed on the entity. If the experiment gives us the expected outcome, we will say that the answer to the test is 'yes.' If the experiment does not give us the expected outcome, we will say that the answer to the test is 'no.' Hence, to define a test, one has to give (1) the measuring apparatus used to perform the experiment connected to the test, (2) the manual of operation of the apparatus, and (3) a rule that allows us to interpret the results in terms of 'yes' and 'no.' Let us denote a well-defined set of tests for an entity S by Q.

We will say that a test α of the entity S in a state p is 'true,' and the corresponding property a is 'actual,' if we can predict with certainty that the answer 'yes' would come out if we were to perform the test.

The way we have introduced the concepts of state, property, test, 'true' test, and 'actual' property has up to now been equivalent to the way they are introduced in the founding papers. As we have remarked, we want to make an explicit distinction between the physical content of the theory and its mathematical form. That is the reason we introduce some additional concepts now.

For a state p we consider the set $\eta(p)$ of all tests $\alpha \in Q$ which are 'true' if the entity is in state p. Let us give now a formal definition of an entity described by its states and its set of testable properties.

Definition 1 (state test entity). A state test entity S is defined by a set Σ (the set of states), a set Q (the set of tests), and a function

$$\eta: \quad \Sigma \to \mathcal{P}(Q): \quad p \mapsto \eta(p) \tag{1}$$

where $\eta(p)$ is, by definition, the set of tests which are 'true' if the entity S is in state p. We call η the state test function. Hence, for a test $\alpha \in Q$ and a state $p \in \Sigma$ we have

$$\alpha \text{ is true if } S \text{ is in state } p \Leftrightarrow \alpha \in \eta(p) \tag{2}$$

We denote a state test entity S as $S(\Sigma, Q, \eta)$.

If the situation is such that whenever the entity S is in a state such that test α is 'true,' then also test β is 'true,' we say that α 'implies' β (or α 'is stronger' than β) and we denote $\alpha < \beta$. Let us now formally introduce the 'implication' on the set of tests of a state test entity.

Definition 2 (test implication). Consider a state test entity $S(\Sigma, Q, \eta)$. For $\alpha, \beta \in Q$ we define

$$\alpha < \beta \Leftrightarrow \text{if for } p \in \Sigma \text{ we have } \alpha \in \eta(p) \text{ then } \beta \in \eta(p) \tag{3}$$

and we say that α 'implies' β and call this relation the 'test implication.'

We have a natural implication on the states that was not identified properly in the founding papers. If for two states $p, q \in \Sigma$ the set $\eta(p)$ of all tests that are 'true' if the entity is in state p includes the set $\eta(q)$ of all tests that are 'true' if the entity is in state q, we say that p implies q (or p is stronger than q) and we write $p < q$.

Definition 3 (state implication). Consider a state test entity $S(\Sigma, Q, \eta)$. For $p, q \in \Sigma$ we define

$$p < q \Leftrightarrow \eta(q) \subset \eta(p) \tag{4}$$

and we say that p 'implies' q and call this relation the 'state implication.'

Proposition 1. Consider a state test entity $S(\Sigma, Q, \eta)$. The implications on Q and Σ are preorder relations. ∎

For a non empty family of tests $(\alpha_i)_i$ we operationally introduce a product test $\Pi_i\alpha_i$, as in the founding papers. It consists in choosing any one of the α_i, performing this chosen test, and considering the outcome obtained as the outcome of $\Pi_i\alpha_i$. We clearly have that $\Pi_i\alpha_i$ is true if and only if α_i is true for each i. This means that $\Pi_i\alpha_i \in \eta(p)$ if and only if $\alpha_i \in \eta(p)\ \forall i$. Let us introduce the concept of 'product test' formally.

Definition 4 (product test). Consider a state test entity $S(\Sigma, Q, \eta)$ and a set $(\alpha_i)_i \in Q$ of tests. A product test $\Pi_i\alpha_i$ is a test such that

$$\Pi_i\alpha_i \in \eta(p) \Leftrightarrow \alpha_i \in \eta(p) \quad \forall i \tag{5}$$

We remark that the notation $\Pi_i\alpha_i$ for a product test is somewhat misleading. Indeed, in general a product test $\Pi_i\alpha_i$ does not have to be a test 'formed' by the α_i, as it is the case in the physical example that inspired the formal definition. It is just a test that satisfies the requirement expressed in formula (5). We remark that this mathematical definition of a product test makes sense for an empty family. In that case, it becomes a test which is always true. This type of test will be formally defined a little later.

Proposition 2. Suppose that we have a state test entity $S(\Sigma, Q, \eta)$. If an arbitrary family of tests $(\alpha_i)_i \in Q$ has a product test $\Pi_i\alpha_i \in Q$; then this product test is an infimum of the $(\alpha_i)_i$ in $Q, <$.

Proof. We clearly have $\Pi_i\alpha_i < \alpha_j\ \forall j$. Suppose that $\beta < \alpha_i\ \forall i$, and consider a state $p \in \Sigma$ such that $\beta \in \eta(p)$. Then we have $\alpha_i \in \eta(p)\ \forall i$. As a consequence we have $\Pi_i\alpha_i \in \eta(p)$. This shows that $\beta < \Pi_i\alpha_i$. ∎

We can define the following test: We do anything that we want with the entity and just give the answer 'yes.' Clearly this test is always 'true.' We can also introduce the following test: We do anything with the entity and just give the answer 'yes' or 'no' as we wish. Clearly this test is never 'true.' Let us define these special types of tests formally.

Definition 5 (unit and zero tests). Consider a state test entity $S(\Sigma, Q, \eta)$. We say that a test τ is a unit test if $\tau \in \eta(p)\ \forall p \in \Sigma$. We say that a test δ is a zero test if $\delta \notin \eta(p)\ \forall p \in \Sigma$.

Proposition 3. Consider a state test entity $S(\Sigma, Q, \eta)$. If τ is a unit test we have for $\alpha \in Q$ that $\alpha < \tau$. If δ is a zero test we have for $\alpha \in Q$ that $\delta < \alpha$. ■

In the founding papers it is supposed that for each nonempty family of tests $(\alpha_i)_i \in Q$ there is a product test $\Pi_i\alpha_i \in Q$. It is also supposed that there exists a unit test $\tau \in Q$ and a zero test $\delta \in Q$. Let us introduce these requirements on the formal level.

Definition 6 (unital product entity). Suppose that we have a state test entity $S(\Sigma, Q, \eta)$. We say that the entity is a 'unital product' entity if Q contains a unit test τ, and a zero test δ and if for each family $(\alpha_i)_i \in Q$ there is a product test $\Pi_i\alpha_i \in Q$.

We remark that, since a product test of the empty family is an always-true test, demanding the existence of a unit test is in fact redundant.

Proposition 4. Consider a unital product entity $S(\Sigma, Q, \eta)$. Then for each set $(\alpha_i)_i \in Q$ of tests there exists an infimum and a supremum for the preorder relation on Q. Further we have, for each unit test τ and zero test δ, and for a set of tests $(\alpha_i)_i$, and $p \in \Sigma$,

$$\tau \in \eta(p) \tag{6}$$

$$\delta \notin \eta(p) \tag{7}$$

$$\alpha_i \in \eta(p)\ \forall i \Leftrightarrow \Pi_i\alpha_i \in \eta(p) \tag{8}$$

and for $p, q \in \Sigma$ and $\alpha, \beta \in Q$ we have

$$p < q \Leftrightarrow \eta(q) \subset \eta(p) \tag{9}$$

$$\alpha < \beta \Leftrightarrow \forall r \in \Sigma\text{: } \alpha \in \eta(r) \text{ then } \beta \in \eta(r) \tag{10}$$

Proof. An infimum for the set $(\alpha_i)_i$ is a product test $\Pi_i\alpha_i$ as we have shown in Proposition 2. It is also easy to see that a product test $\Pi_{\{\alpha_i < \beta \forall i\}}\beta$ is a supremum for the family $(\alpha_i)_i$. ■

In general there is no *a priori* correspondence between properties and tests. Some properties can be tested and some tests give rise to properties. We have discussed this general situation in detail in Aerts (1998) and will not repeat it here. In fact here, as was also the case in the founding papers, we are interested in the situation where we consider only testable properties.

And we will, as in the founding papers, define properties as the equivalence classes of tests.

Definition 7. Consider a state test entity $S(\Sigma, Q, \eta)$. Two tests $\alpha, \beta \in Q$ are said to be 'equivalent,' $\alpha \approx \beta$, if both $\alpha < \beta$ and $\beta < \alpha$ hold. In other words, $\alpha \approx \beta$ iff for $p \in \Sigma$, $\alpha \in \eta(p) \Leftrightarrow \beta \in \eta(p)$.

If α and β are equivalent, they are considered to test the same property. That is why we will identify the properties of the entity with the equivalence classes of tests.

Definition 8 (property). Consider a state test entity $S(\Sigma, Q, \eta)$. Let $\alpha \in Q$ be a test. The 'property' $a(\alpha)$ tested by α is defined to be the equivalence class of α in $Q/\approx$. In other words,

$$a(\alpha) = \{\beta \in Q | \beta \approx \alpha\} \tag{11}$$

The set of all properties of the entity will be denoted $\mathcal{L}$, i.e., $\mathcal{L} = Q/\approx$.

For the description of an entity by means of its states and properties we propose state property systems, which were first defined in Aerts (1998). We show that a unital product entity gives rise to a state property system.

Definition 9. We say that $(\Sigma, <, \mathcal{L}, <, \wedge, \vee, \xi)$, or more concisely $(\Sigma, \mathcal{L}, \xi)$, is a 'state property system' if $(\Sigma, <)$ is a preordered set, $(\mathcal{L}, <, \wedge, \vee)$ is a complete lattice, and ξ is a function

$$\xi: \quad \Sigma \to \mathcal{P}(\mathcal{L}) \tag{12}$$

such that for $p \in \Sigma$, I the maximal element, 0 the minimal element of $\mathcal{L}$, and $(a_i)_i \in \mathcal{L}$, we have

$$I \in \xi(p) \tag{13}$$

$$0 \notin \xi(p) \tag{14}$$

$$a_i \in \xi(p)\ \forall i \Leftrightarrow \wedge_i\, a_i \in \xi(p) \tag{15}$$

and for $p, q \in \Sigma$ and $a, b \in \mathcal{L}$ we have

$$p < q \Leftrightarrow \xi(q) \subset \xi(p) \tag{16}$$

$$a < b \Leftrightarrow \forall r \in \Sigma: a \in \xi(r) \text{ then } b \in \xi(r) \tag{17}$$

We remark that the reverse arrow of (15) follows from (17) and hence could be left out of the definition. Indeed, we clearly have $\wedge_i\, a_i < a_j\ \forall j$, which means that $\forall p \in \Sigma: \wedge_i a_i \in \xi(p) \Rightarrow a_j \in \xi(p)\ \forall j$.

Theorem 1. Consider a unital product entity $S(\Sigma, Q, \eta)$. The triple $(\Sigma, \mathcal{L}, \xi)$ where

$$\mathcal{L} = \{a(\alpha)|\alpha \in Q\} \tag{18}$$

is partially ordered by

$$a(\alpha) < a(\beta) \Leftrightarrow \alpha < \beta \qquad (\alpha, \beta \in Q) \tag{19}$$

and ξ is the following function:

$$\xi: \quad \Sigma \to \mathcal{P}(\mathcal{L}) \tag{20}$$

$$p \mapsto \xi(p) = \{a(\alpha)|\alpha \in \eta(p)\} \tag{21}$$

is a state property system. The top and bottom element of $\mathcal{L}$ are given respectively by

$$I = a(\tau) \tag{22}$$

$$0 = a(\delta) \tag{23}$$

where τ is a unit test and δ is a zero test.

Proof. Let us prove that $\mathcal{L}$ is a complete lattice. The relation $<$ on $\mathcal{L}$ is well defined: $\alpha' \approx \alpha < \beta \approx \beta' \Rightarrow \alpha' < \beta'$. We clearly have that $(\mathcal{L}, <)$ is a preordered set. We prove that $<$ is also antisymmetric. Consider two properties $b, c \in \mathcal{L}$ such that $b < c$ and $c < b$. Then there exists $\varepsilon, \gamma \in Q$ such that $b = a(\varepsilon)$ and $c = a(\gamma)$. Now, $a(\varepsilon) < a(\gamma)$ implies that $\varepsilon < \gamma$ and $a(\gamma) < a(\varepsilon)$ implies that $\gamma < \varepsilon$. This means that ε and γ are equivalent. Consequently $a(\varepsilon) = a(\gamma)$. This shows that $(\mathcal{L}, <)$ is a partially ordered set.

Consider now an arbitrary set $(a_i)_i \in \mathcal{L}$. Then there exists a set $(\alpha_i)_i \in Q$ such that $a_i = a(\alpha_i)\ \forall i$. Consider a product test $\Pi_i\alpha_i$. Then $a(\Pi_i\alpha_i)$ is the infimum of the set $(a_i)_i$. Indeed, consider a state $p \in \Sigma$ such that $a(\Pi_i\alpha_i) \in \xi(p)$. Consequently, $\alpha_i \in \eta(p)\ \forall i$. So we have $a(\alpha_i) \in \xi(p)\ \forall i$. This shows that $a(\Pi_i\alpha_i) < a(\alpha_j)\ \forall j$. Suppose now that $a(\gamma) < a(\alpha_i)\ \forall i$ with $\gamma \in Q$ and consider a state $p \in \Sigma$ such that $a(\gamma) \in \xi(p)$. Then we have $a(\alpha_i) \in \xi(p)$ $\forall i$. Consequently we have that $\alpha_i \in \eta(p)\ \forall i$. This implies that $\Pi_i\alpha_i \in \eta(p)$ and so we have $a(\Pi_i\alpha_i) \in \xi(p)$. This shows that $a(\gamma) < a(\Pi_i\alpha_i)$. Therefore $\mathcal{L}$ has arbitrary infima. It follows (and this is a result due to Birkhoff) that $\mathcal{L}$ has arbitrary suprema: for $(a_i)_i \in \mathcal{L}$: $\vee_i a_i = \wedge \{b \in \mathcal{L}|a_i < b\ \forall i\}$. So $(\mathcal{L}, <)$ is a complete lattice.

For a unit test τ and a state p we have that $\tau \in \eta(p)$. Consequently $I = a(\tau) \in \xi(p)$. For a zero test δ and a state p we have that $\delta \notin \eta(p)$. This implies that $0 = a(\delta) \notin \xi(p)$.

Next we verify (15). Consider $(a_i)_i \in \mathcal{L}$ and a state p such that $a_i \in \xi(p)\ \forall i$. Since there exists a set $(\alpha_i)_i \in Q$ such that $a_i = a(\alpha_i)\ \forall i$, we have that $\alpha_i \in \eta(p)\ \forall i$. Consequently $\Pi_i\alpha_i \in \eta(p)$. This implies that $a(\Pi_i\alpha_i) \in \xi(p)$. So we have that $\wedge_i a_i = \wedge_i a(\alpha_i) = a(\Pi_i\alpha_i) \in \xi(p)$.

Equations (16) and (17) are easily verified. ■

3. STATE PROPERTY SYSTEMS AND CLOSURE SPACES

In this section we will investigate the state property systems and show that they are in natural correspondence with closure spaces.

Proposition 5. Suppose that $(\Sigma, \mathcal{L}, \xi)$ is a state property system. We introduce the 'Cartan map' κ:

$$\kappa: \quad \mathcal{L} \to \mathcal{P}(\Sigma): \quad a \mapsto \kappa(a) = \{p \in \Sigma | a \in \xi(p)\} \tag{24}$$

For a, b, $(a_i)_i \in \mathcal{L}$ we have

$$\kappa(I) = \Sigma \tag{25}$$

$$\kappa(0) = \emptyset \tag{26}$$

$$a < b \Leftrightarrow \kappa(a) \subset \kappa(b) \tag{27}$$

$$\kappa(\wedge_i a_i) = \bigcap_i \kappa(a_i) \tag{28}$$

It follows that $\kappa: \mathcal{L} \to (\kappa(\mathcal{L}), \subset, \cap)$ is an isomorphism of complete lattices.

Proof. Since $I \in \xi(p)\ \forall p \in \Sigma$, we have $\kappa(I) = \Sigma$. Since $0 \notin \xi(p)\ \forall p \in \Sigma$ we have $\kappa(0) = \emptyset$. To prove (27), just remark that (17) can be rewritten as

$$a < b \Leftrightarrow \forall r \in \Sigma: r \in \kappa(a) \text{ then } r \in \kappa(b) \tag{29}$$

From $\wedge_i a_i < a_j\ \forall j$ it follows that $\kappa(\wedge_i a_i) \subset \kappa(a_j)\ \forall j$. This yields $\kappa(\wedge_i a_i) \subset \cap_i \kappa(a_i)$. To prove the other inclusion, take $p \in \cap_i \kappa(a_i)$; then $p \in \kappa(a_j)\ \forall j$. Hence $a_j \in \xi(p)\ \forall j$, which implies, by (15), that $\wedge_i a_i \in \xi(p)$. From this it follows that $p \in \kappa(\wedge_i a_i)$. As a consequence we have $\cap_i \kappa(a_i) \subset \kappa(\wedge_i a_i)$. ■

To avoid misunderstandings we recall the definition of a closure space.

Definition 10. A 'closure space' $(Z, \mathcal{G})$ consists of a set Z and a family of subsets $\mathcal{G} \subset \mathcal{P}(Z)$ satisfying the following conditions:

$$Z \in \mathcal{G}, \qquad \emptyset \in \mathcal{G} \tag{30}$$

$$(G_i)_i \in \mathcal{G} \Rightarrow \cap_i G_i \in \mathcal{G} \tag{31}$$

If these conditions hold, we call $\mathcal{G}$ a 'closure system' on Z. The 'closure operator' corresponding to this closure space is defined as

$$cl: \quad \mathcal{P}(Z) \to \mathcal{P}(Z): \quad Y \mapsto \cap\{G \in \mathcal{G} | Y \subset G\} \tag{32}$$

Theorem 2. Suppose that $(\Sigma, \mathcal{L}, \xi)$ is a state property system. Let us introduce

$$\mathcal{F} = \kappa(\mathcal{L}) = \{\kappa(a) | a \in \mathcal{L}\} \tag{33}$$

Then $\mathcal{F}$ is a closure system on Σ.

Proof. From the foregoing proposition it follows that $\Sigma \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$. Consider $(F_i)_i \in \mathcal{F}$. Then there exists $(a_i)_i \in \mathcal{L}$ such that $\kappa(a_i) = F_i \ \forall i$. We have $\kappa(\wedge_i a_i) = \cap_i \kappa(a_i) = \cap_i F_i$. This shows that $\cap_i F_i \in \mathcal{F}$. ■

This theorem shows that to a state property system there naturally corresponds a closure system on the set of states, where the properties are represented by the closed subsets. We can also associate a state property system with any closure space.

Theorem 3. Consider a closure space $(\Sigma, \mathcal{F})$. We introduce the following definitions for $F, G, (F_i)_i \in \mathcal{F}$ and $p, q \in \Sigma$:

$$F < G \Leftrightarrow F \subset G \tag{34}$$

$$\wedge_i F_i = \cap_i F_i \tag{35}$$

$$\vee_i F_i = cl(\cup_i F_i) \tag{36}$$

$$\xi: \quad \Sigma \rightarrow \mathcal{P}(\mathcal{F}): \quad p \mapsto \{F \in \mathcal{F} | p \in F\} \tag{37}$$

$$p < q \Leftrightarrow \xi(q) \subset \xi(p) \tag{38}$$

Then $(\Sigma, <, \mathcal{F}, <, \wedge, \vee, \xi)$ is a state property system.

Proof. It is easy to show that $(\mathcal{F}, <, \wedge, \vee)$ is a complete lattice, with maximal element $I = \Sigma$ and minimal element $0 = \emptyset$. It is trivial to verify that (38) defines a preorder on Σ. Clearly, we have $I \in \xi(p)$, $0 \notin \xi(p)$ $\forall p \in \Sigma$. Next, suppose that $F_i \in \xi(p) \ \forall i$. This means that $p \in F_i \ \forall i$ or $p \in \cap_i F_i$. As a consequence we have $\wedge_i F_i = \cap_i F_i \in \xi(p)$. Finally we verify (17). Let $F, G \in \mathcal{L}$. We then have $F < G \Leftrightarrow F \subset G \Leftrightarrow (p \in F \Rightarrow p \in G) \Leftrightarrow (F \in \xi(p) \Rightarrow G \in \xi(p))$ and we are done. ■

4. THE MORPHISMS

Theorems 2 and 3 show that there is a straightforward correspondence between state property systems and closure spaces. We can extend this correspondence to "natural" morphisms of these two structures. In this section we introduce morphisms of state property systems and show their connection to continuous maps between closure spaces.

Consider two state property systems $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$. As explained in Section 2, these state property systems respectively describe entities S and S'. We will arrive at the notion of morphism by analyzing the situation where the entity S is a subentity of the entity S'. In that case, the following three natural requirements should be satisfied:

(i) If the entity S' is in a state p', then the state $m(p')$ of S is determined. This defines a function m from the set of states of S' to the set of states of S.

(ii) If we consider a property a of the entity S, then to a there corresponds a property $n(a)$ of the "bigger" entity S'. This defines a function n from the set of properties of S to the set of properties of S'.

(iii) We want a and $n(a)$ to be two descriptions of the "same" property of S, once considered as an entity in itself, once as a subentity of S'. In other words we want a and $n(a)$ to be actual at once. This means that for a state p' of S' [and a corresponding state $m(p')$ of S] we want the following "covariance principle" to hold:

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{39}$$

We are now ready to present a formal definition of a morphism of state property systems.

Definition 11. Consider two state property systems $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$. We say that

$$(m, n)\colon \quad (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi) \tag{40}$$

is a 'morphism' (of state property systems) if m is a function

$$m\colon \quad \Sigma' \to \Sigma \tag{41}$$

and n is a function

$$n\colon \quad \mathcal{L} \to \mathcal{L}' \tag{42}$$

such that for $a \in \mathcal{L}$ and $p' \in \Sigma'$ the following holds:

$$a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \tag{43}$$

The following is an elegant rewriting of this definition.

Proposition 6. Consider two state property systems $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$. Two functions $m\colon \Sigma' \to \Sigma$ and $n\colon \mathcal{L} \to \mathcal{L}'$ define a morphism $(m, n)\colon (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$ if and only if we have

$$\xi \circ m = n^{-1} \circ \xi' \tag{44}$$

where $n^{-1}\colon \mathcal{P}(\mathcal{L}') \to \mathcal{P}(\mathcal{L})\colon F' \mapsto n^{-1}(F') = \{a \in \mathcal{L} | n(a) \in F'\}$. ■

The next proposition gives some properties of morphisms.

Proposition 7. Consider two state property systems $(\Sigma, \mathcal{L}, \xi)$ and $(\Sigma', \mathcal{L}', \xi')$ connected by a morphism $(m, n)\colon (\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$. For $p', q' \in \Sigma'$ and $a, b, (a_i)_i \in \mathcal{L}$ we have

$$p' < q' \Rightarrow m(p') < m(q') \tag{45}$$

$$a < b \Rightarrow n(a) < n(b) \tag{46}$$

$$n(\wedge_i a_i) = \wedge_i n(a_i) \tag{47}$$

$$n(I) = I' \tag{48}$$

$$n(0) = 0' \tag{49}$$

Proof. Suppose that $p' < q'$. We then have $\xi'(q') \subset \xi'(p')$. From this it follows that $n^{-1}(\xi'(q')) \subset n^{-1}(\xi'(p'))$. Through (44) this yields $\xi(m(q')) \subset \xi(m(p'))$, whence $m(p') < m(q')$.

Next consider $a < b$ and let $r' \in \Sigma'$ be such that $n(a) \in \xi'(r')$. Then we have $a \in \xi(m(r'))$ and, since $a < b$, this yields $b \in \xi(m(r'))$. From this it follows that $n(b) \in \xi'(r')$. So we have shown that $n(a) < n(b)$.

From $\wedge_i a_i < a_j \ \forall j$ we obtain $n(\wedge_i a_i) < n(a_j) \ \forall j$. This yields $n(\wedge_i a_i) < \wedge_i n(a_i)$. We still have to show that $\wedge_i n(a_i) < n(\wedge_i a_i)$. Let $r' \in \Sigma'$ be such that $\wedge_i n(a_i) \in \xi'(r')$. This implies that $n(a_j) \in \xi'(r') \ \forall j$ [use (17)]. But from this we obtain $a_j \in \xi(m(r')) \ \forall j$ and hence $\wedge_i a_i \in \xi(m(r'))$. As a consequence we have $n(\wedge_i a_i) \in \xi'(r')$. But then we have shown that $\wedge_i n(a_i) < n(\wedge_i a_i)$.

We clearly have $n(I) < I'$. For all $r' \in \Sigma'$, we have $I \in \xi(m(r'))$ and hence $n(I) \in \xi'(r')$. Through (17) this implies $I' < n(I)$, whence $n(I) = I'$. Trivially $0' < n(0)$. Suppose $n(0) < 0'$ does not hold. Then the contraposition of (17) says there is an $r' \in \Sigma'$ such that $n(0) \in \xi'(r')$. This would imply $0 \in \xi(m(r'))$, which is impossible. Therefore we have proven $n(0) = 0'$. ■

Proposition 8. Suppose that we have a morphism of state property systems (m, n): $(\Sigma', \mathscr{L}', \xi') \to (\Sigma, \mathscr{L}, \xi)$. Consider the Cartan maps κ and κ' that connect these state property systems to their corresponding closure spaces $(\Sigma, \mathscr{F})$ and $(\Sigma', \mathscr{F}')$, as was done in Theorem 2. For $a \in \mathscr{L}$ we have

$$m^{-1}(\kappa(a)) = \kappa'(n(a)) \tag{50}$$

Proof. We have $p' \in m^{-1}(\kappa(a)) \Leftrightarrow m(p') \in \kappa(a) \Leftrightarrow a \in \xi(m(p')) \Leftrightarrow n(a) \in \xi'(p') \Leftrightarrow p' \in \kappa'(n(a))$. ■

We can now connect morphisms of state property systems to continuous maps (morphisms of closure spaces).

Proposition 9. Suppose that we have a morphism of state property systems (m, n): $(\Sigma', \mathscr{L}', \xi') \to (\Sigma, \mathscr{L}, \xi)$. If $(\Sigma, \mathscr{F})$ and $(\Sigma', \mathscr{F}')$ are the closure spaces corresponding to these state property systems (cf. Theorem 2), then m: $(\Sigma', \mathscr{F}') \to (\Sigma, \mathscr{F})$ is continuous.

Proof. Take a closed subset $F \in \mathscr{F}$. Then there is an $a \in \mathscr{L}$ such that $\kappa(a) = F$. From the foregoing proposition we have $m^{-1}(F) = m^{-1}(\kappa(a)) = \kappa'(n(a)) \in \mathscr{F}'$. This proves our claim. ■

Proposition 10. Suppose we have two closure spaces $(\Sigma, \mathcal{F})$ and $(\Sigma', \mathcal{F}')$ and a continuous map m: $\Sigma' \to \Sigma$. Consider the state property systems $(\Sigma, \mathcal{F}, \xi)$ and $(\Sigma', \mathcal{F}', \xi')$ corresponding to these two closure systems, as proposed in Theorem 3. Then (m, m^{-1}) is a morphism from $(\Sigma', \mathcal{F}', \xi')$ to $(\Sigma, \mathcal{F}, \xi)$.

Proof. Continuity yields that m^{-1} is a function from $\mathcal{F}$ to $\mathcal{F}'$. Let us now show formula (43) using the definition of ξ' and ξ as put forward in Theorem 3. For $F \in \mathcal{F}$ and $p' \in \Sigma'$ we have $F \in \xi(m(p')) \Leftrightarrow m(p') \in F \Leftrightarrow p \in m^{-1}(F) \Leftrightarrow m^{-1}(F) \in \xi'(p')$. ∎

5. AN EQUIVALENCE OF CATEGORIES

The previous section demonstrates that there is a strong connection between state property systems with their morphisms and closure spaces with continuous maps. In this section we formalize this connection into an equivalence of categories. We suppose that the reader is familiar with basic category theory and refer the reader who is not to Borceux (1994).

We will introduce the categories, but before doing so, we define the composition of morphisms of state property systems.

Definition 12. Given two morphisms of state property systems (m_1, n_1): $(\Sigma_1, \mathcal{L}_1, \xi_1) \to (\Sigma_2, \mathcal{L}_2, \xi_2)$ and (m_2, n_2): $(\Sigma_2, \mathcal{L}_2, \xi_2) \to (\Sigma_3, \mathcal{L}_3, \xi_3)$, their composite is defined as

$$(m_2, n_2) \circ (m_1, n_1) = (m_2 \circ m_1, n_1 \circ n_2) \tag{51}$$

Proposition 11. Given two morphisms of state property systems (m_1, n_1): $(\Sigma_1, \mathcal{L}_1, \xi_1) \to (\Sigma_2, \mathcal{L}_2, \xi_2)$ and (m_2, n_2): $(\Sigma_2, \mathcal{L}_2, \xi_2) \to (\Sigma_3, \mathcal{L}_3, \xi_3)$, their composite $(m_2, n_2)\circ(m_1, n_1)$: $(\Sigma_1, \mathcal{L}_1, \xi_1) \to (\Sigma_3, \mathcal{L}_3, \xi_3)$ is again a morphism of state property systems.

Proof. We prove our claim by checking formula (44). We have $\xi_3 \circ (m_2 \circ m_1) = (\xi_3 \circ m_2) \circ m_1 = (n_2^{-1} \circ \xi_2) \circ m_1 = n_2^{-1} \circ (\xi_2 \circ m_1) = n_2^{-1} \circ (n_1^{-1} \circ \xi_1) = (n_1 \circ n_2)^{-1} \circ \xi_1$, which proves the assertion. ∎

Proposition 12. The composition of morphisms of state property systems is associative, and given a morphism (m, n): $(\Sigma', \mathcal{L}', \xi') \to (\Sigma, \mathcal{L}, \xi)$, the following equalities hold:

$$(m, n) \circ (id_{\Sigma'}, id_{\mathcal{L}'}) = (m, n) \tag{52}$$

$$(id_{\Sigma}, id_{\mathcal{L}}) \circ (m, n) = (m, n) \tag{53}$$

Having these results under our belt, we can safely state the following definitions.

Definition 13. We call **SP** the category of state property systems (Definition 9) with their morphisms (Definition 11) and **Cls** is the category of closure spaces (Definition 10) with continuous maps.

Let us introduce the functors which will establish the equivalence of categories.

Theorem 4. The correspondence F: **SP** $\to$ **Cls** consisting of (1) the mapping

$$|\mathbf{SP}| \to |\mathbf{Cls}| \tag{54}$$

$$(\Sigma, \mathscr{L}, \xi) \mapsto F(\Sigma, \mathscr{L}, \xi) \tag{55}$$

where $F(\Sigma, \mathscr{L}, \xi)$ is the closure space $(\Sigma, \mathscr{F})$ given by Theorem 2; and (2) for every pair of objects $(\Sigma, \mathscr{L}, \xi)$, $(\Sigma', \mathscr{L}', \xi')$ of **SP** the mapping

$$\mathbf{SP}((\Sigma', \mathscr{L}', \xi'), (\Sigma, \mathscr{L}, \xi)) \to \mathbf{Cls}(F(\Sigma', \mathscr{L}', \xi'), F(\Sigma, \mathscr{L}, \xi)) \tag{56}$$

$$(m, n) \mapsto m \tag{57}$$

is a covariant functor.

Proof. This is, apart from some minor checks, a consequence of Theorem 2 and Proposition 9. ■

Theorem 5. The correspondence G: **Cls** $\to$ **SP** consisting of (1) the mapping

$$|\mathbf{Cls}| \to |\mathbf{SP}| \tag{58}$$

$$(\Sigma, \mathscr{F}) \mapsto G(\Sigma, \mathscr{F}) \tag{59}$$

where $G(\Sigma, \mathscr{F})$ is the state property system $(\Sigma, \mathscr{F}, \xi)$ given by Theorem 3; and (2) for every pair of objects $(\Sigma, \mathscr{F})$, $(\Sigma', \mathscr{F}')$ of **Cls** the mapping

$$\mathbf{Cls}((\Sigma', \mathscr{F}'), (\Sigma, \mathscr{F})) \to \mathbf{SP}(G(\Sigma', \mathscr{F}'), G(\Sigma, \mathscr{F})) \tag{60}$$

$$m \mapsto (m, m^{-1}) \tag{61}$$

is a covariant functor.

Proof. This is, apart from some minor checks, a consequence of Theorem 3 and Proposition 10. ■

Next we characterize the isomorphisms of **SP**.

Proposition 13. A morphism $(m, n) \in \mathbf{SP}((\Sigma', \mathscr{L}', \xi'), (\Sigma, \mathscr{L}, \xi))$ is an isomorphism if and only if m: $\Sigma' \to \Sigma$ and n: $\mathscr{L} \to \mathscr{L}'$ are bijective.

Proof. Let (m, n) be an isomorphism. The fact that it has a right inverse implies that m is surjective and that n is injective. On the other hand, we conclude from the existence of a left inverse that m is injective and that n is surjective.

Now, let (m, n) be a morphism with m and n bijective. Let $m^{-1}: \Sigma \to \Sigma'$ and $n^{-1}: \mathcal{L}' \to \mathcal{L}$ be the inverses of m and n. We show that (m^{-1}, n^{-1}) is a morphism, using (44). From $\xi \circ m = n^{-1} \circ \xi'$ we obtain $\xi' = n \circ \xi \circ m$, where $n^{-1}: \mathcal{P}(\mathcal{L}') \to \mathcal{P}(\mathcal{L})$ and $n: \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L}'): T \mapsto n(T) = \{n(a) | a \in T\}$. This implies $\xi' \circ m^{-1} = n \circ \xi \circ m \circ m^{-1} = n \circ \xi$, which proves our assertion. ■

We have arrived at the following result.

Theorem 6 (equivalence of ***SP*** *and* ***Cls***). The functors F: **SP** $\to$ **Cls** and G: **Cls** $\to$ **SP** establish an equivalence of categories. Moreover, $F \circ G = Id_{\mathbf{Cls}}$.

Proof. Step 1: $G \circ F$. Given an object $(\Sigma, \mathcal{L}, \xi) \in |\mathbf{SP}|$, we have $GF(\Sigma, \mathcal{L}, \xi) = (\Sigma, \kappa(\mathcal{L}), \bar{\xi})$, where $\kappa: \mathcal{L} \to \mathcal{P}(\Sigma)$ is the Cartan map defined in Proposition 5 and

$$\bar{\xi}: \Sigma \to \mathcal{P}(\kappa(\mathcal{L})) \tag{62}$$

$$p \mapsto \{\kappa(a) | a \in \mathcal{L}, p \in \kappa(a)\} = \{\kappa(a) | a \in \xi(p)\} \tag{63}$$

Given a morphism $(m, n) \in \mathbf{SP}((\Sigma', \mathcal{L}', \xi'), (\Sigma, \mathcal{L}, \xi))$, we obtain $GF(m, n) = (m, m^{-1}) \in \mathbf{SP}((\Sigma', \kappa'(\mathcal{L}'), \bar{\xi}'), (\Sigma, \kappa(\mathcal{L}), \bar{\xi}))$

Step 2: $GF \cong Id_{\mathbf{SP}}$. For any object $(\Sigma, \mathcal{L}, \xi) \in |\mathbf{SP}|$, define

$$\varepsilon_{(\Sigma, \mathcal{L}, \xi)}: GF(\Sigma, \mathcal{L}, \xi) \to (\Sigma, \mathcal{L}, \xi) \tag{64}$$

$$\varepsilon_{(\Sigma, \mathcal{L}, \xi)} = (id_\Sigma, \kappa) \tag{65}$$

Then $\varepsilon = (\varepsilon_{(\Sigma, \mathcal{L}, \xi)}): GF \to Id_{\mathbf{SP}}$ is a natural isomorphism. First we verify that $\varepsilon_{(\Sigma, \mathcal{L}, \xi)}$ is a morphism of **SP**. Indeed, for $a \in \mathcal{L}, p \in \Sigma$ we have $\kappa(a) \in \bar{\xi}(p) \Leftrightarrow p \in \kappa(a) \Leftrightarrow a \in \xi(p) = \xi(id_\Sigma p)$. To show that (id_Σ, κ) is an isomorphism, we only have to prove that $\kappa: \mathcal{L} \to \kappa(\mathcal{L})$ is bijective (Proposition 13) and this follows from Proposition 5. The naturality of ε is an immediate consequence of Proposition 8.

Step 3: $FG = Id_{\mathbf{Cls}}$. For the morphisms this is trivial:

$$m \overset{G}{\mapsto} (m, m^{-1}) \overset{F}{\mapsto} m \tag{66}$$

where m is a morphism of **Cls**. Now consider an arbitrary closure space $(\Sigma, \mathcal{F})$. Then $G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \xi)$ where $\xi(p) = \{F \in \mathcal{F} | p \in F\}$. Hence the corresponding Cartan map is given by

$$\kappa\colon \mathcal{F} \to \mathcal{P}(\Sigma) \tag{67}$$

$$F \mapsto \{p \in \Sigma | F \in \xi(p)\} = \{p \in \Sigma | p \in F\} = F \tag{68}$$

This implies $FG(\Sigma, \mathcal{F}) = F(\Sigma, \mathcal{F}, \xi) = (\Sigma, \kappa(\mathcal{F})) = (\Sigma, \mathcal{F})$. ■

6. THE FIRST AXIOM: STATE DETERMINATION AND T_0 SEPARATION

In the founding papers the state $p \in \Sigma$ of an entity S is identified with the set of all properties $a \in \mathcal{L}$ it makes actual. In this section, we investigate the consequences this assumption has on state property systems and closure spaces.

Let $S(\Sigma, Q, \eta)$ be a unital product entity and let $(\Sigma, \mathcal{L}, \xi)$ be its state property system. Remember that for a state $p \in \Sigma$ we have that

$$\xi(p) = \{a \in \mathcal{L} | a \text{ is actual when } S \text{ is in state } p\} \tag{69}$$

Hence the demand that a state p be completely determined by the set of all properties it makes actual, i.e., by $\xi(p)$, is mathematically expressed by

"$\xi\colon \quad \Sigma \to \mathcal{P}(\mathcal{L})$ is injective"

Definition 14. A closure space $(Z, \mathcal{G})$ is called 'T_0' if for $x,y \in Z$ we have $cl(x) = cl(y) \Rightarrow x = y$, where $cl(x)$ is the usual notation for $cl(\{x\})$.

Let us give some equivalent conditions to the injectivity of ξ.

Proposition 14. Let $S(\Sigma,Q, \eta)$ be a unital product entity and let $(\Sigma, \mathcal{L}, \xi)$ be the state property system it generates. The following are equivalent:

(1) $\xi\colon \Sigma \to \mathcal{P}(\mathcal{L})$ is injective.

(2) The preorder $<$ on Σ is a partial order.

(3) $\eta\colon \Sigma \to \mathcal{P}(Q)$ is injective.

(4) $(\Sigma, \mathcal{F}) = F(\Sigma, \mathcal{L}, \xi)$ is a T_0 closure space.

Proof. (1 $\Leftrightarrow$ 2) Remember that we have for $p, q \in \Sigma$: $p < q$ iff $\xi(q) \subset \xi(p)$. Hence $p < q$ and $q < p$ is equivalent to $\xi(q) \subset \xi(p)$ and $\xi(p) \subset \xi(q)$ [or $\xi(p) = \xi(q)$]. It follows that the injectivity of ξ is equivalent to the antisymmetry of $<$.

(1 $\Leftrightarrow$ 3) This is an immediate consequence of $\eta(q) \subset \eta(p) \Leftrightarrow p < q \Leftrightarrow \xi(q) \subset \xi(p)$, where the first '$\Leftrightarrow$' is the definition of $<$ and where $p, q \in \Sigma$.

(1 $\Rightarrow$ 4) Suppose $p, q \in \Sigma$ are such that $cl(p) = cl(q)$. From the definition of $(\Sigma, \mathcal{F})$ we have that $cl(p) = \cap_{p \in \kappa(a)} \kappa(a) = \cap_{a \in \xi(p)} \kappa(a)$, where $\kappa\colon \mathcal{L} \to \mathcal{P}(\Sigma)$ is the Cartan map defined in Proposition 5. Hence we have $p \in \cap_{a \in \xi(p)} \kappa(a) = \cap_{a \in \xi(q)} \kappa(a) \ni q$. This yields that $p \in \kappa(a)$ for every $a \in \xi(q)$, or in other words, $a \in \xi(q) \Rightarrow a \in \xi(p)$. This shows

$\xi(q) \subset \xi(p)$. Similarly $q \in \cap_{a\in\xi(p)}\kappa(a)$ gives $\xi(p) \subset \xi(q)$. So we have $\xi(p) = \xi(q)$, whence by (1) $p = q$ holds.

($4 \Rightarrow 1$) Consider $p, q \in \Sigma$ with $\xi(p) = \xi(q)$. Then $cl(p) = \cap_{a\in\xi(p)} \kappa(a) = \cap_{a\in\xi(q)} \kappa(a) = cl(q)$. Since $(\Sigma, \mathcal{F})$ is T_0 (4), we have $p = q$. ■

The following terminology is taken from Aerts (1994).

Definition 15 (state-determined entity). We call a state test entity $S(\Sigma, Q, \eta)$ 'state determined' if $\eta:\Sigma\rightarrow\mathcal{P}(Q)$ is injective. We will call a unital product entity $S(\Sigma, Q, \eta)$ a 'state-determined entity' if it is state determined. A state property system $(\Sigma, \mathcal{L}, \xi)$ is a 'state-determined state property system' if ξ is injective.

Definition 16. We define $\mathbf{SP_0}$ as the subcategory of $\mathbf{SP}$ where the objects are given by

$$|\mathbf{SP_0}| = \{(\Sigma, \mathcal{L}, \xi) \in |\mathbf{SP}|: \xi \text{ is injective}\} \tag{70}$$

and the morphisms by

$$\mathbf{SP_0}((\Sigma', \mathcal{L}', \xi'),(\Sigma, \mathcal{L}, \xi)) = \mathbf{SP}((\Sigma', \mathcal{L}', \xi'),(\Sigma, \mathcal{L}, \xi)) \tag{71}$$

where $(\Sigma, \mathcal{L}, \xi), (\Sigma', \mathcal{L}', \xi') \in \mathbf{SP_0}$. So $\mathbf{SP_0}$ is the category of state-determined state property systems. Similarly we will use $\mathbf{Cls_0}$ for the category of T_0 closure spaces with continuous maps as morphisms.

Clearly $\mathbf{Cls_0}$ is an isomorphism-closed subcategory of $\mathbf{Cls}$. We prove that the same holds for $\mathbf{SP_0}$.

Proposition 15. The category $\mathbf{SP_0}$ is an isomorphism-closed subcategory of $\mathbf{SP}$: if $(\Sigma', \mathcal{L}', \xi') \in \mathbf{SP_0}$, $(\Sigma, \mathcal{L}, \xi) \in \mathbf{SP}$, and (m, n): $(\Sigma', \mathcal{L}', \xi') \rightarrow (\Sigma, \mathcal{L}, \xi)$ is an isomorphism of $\mathbf{SP}$, then (m, n) is an isomorphism of $\mathbf{SP_0}$, in particular $(\Sigma, \mathcal{L}, \xi) \in \mathbf{SP_0}$.

Proof. By equation (71) we only have to show that $(\Sigma, \mathcal{L}, \xi) \in \mathbf{SP_0}$, i.e., that ξ is injective. Suppose $\xi(p) = \xi(q)$ holds for some $p, q \in \Sigma$. Put $p' = m^{-1}(p)$, $q' = m^{-1}(q)$. We show $\xi'(p') = \xi'(q')$. Indeed, $a = n(n^{-1}(a)) \in \xi'(p') \Leftrightarrow n^{-1}(a) \in \xi(m(p')) = \xi(p) = \xi(q) = \xi(m(q')) \Leftrightarrow a = n(n^{-1}(a)) \in \xi'(q')$. Since ξ is injective, this implies $p' = q'$, whence $p = q$. ■

We also have the following:

Proposition 16. Let $(\Sigma, \mathcal{F})$ be a closure space. Let G: $\mathbf{Cls} \rightarrow \mathbf{SP}$ be the functor defined in Theorem 5. Then

$$(\Sigma,\mathcal{F}) \in |\mathbf{Cls_0}| \Leftrightarrow G(\Sigma,\mathcal{F}) = (\Sigma,\mathcal{F},\xi) \in |\mathbf{SP_0}| \tag{72}$$

where, as in Theorem 3, ξ: $\Sigma \rightarrow \mathcal{P}(\mathcal{F})$: $p \mapsto \{F \in \mathcal{F}|p\in F\}$.

Proof: ($\Rightarrow$) Suppose $cl(p) = cl(q)$ implies $p = q$ for all $p,q \in \Sigma$. We have to show that ξ is injective. Suppose $\xi(p) = \xi(q)$ for some p, $q \in \Sigma$. Then $cl(p) = \cap\xi(p) = \cap\xi(q) = cl(q)$. This yields $p = q$.

($\Leftarrow$) If $G(\Sigma, \mathcal{F}) = (\Sigma, \mathcal{F}, \xi) \in |\mathbf{SP_0}|$, then ξ is injective. By Theorem 6 and Proposition 14 $(\Sigma, \mathcal{F}) = FG(\Sigma, \mathcal{F}) \in |\mathbf{Cls_0}|$. ■

We can now prove the following:

Theorem 7 (equivalence of $\mathbf{SP_0}$ and $\mathbf{Cls_0}$). The covariant functors F and G (see Theorems 4 and 5) restrict and corestrict to functors F: $\mathbf{SP_0} \to \mathbf{Cls_0}$ and G: $\mathbf{Cls_0} \to \mathbf{SP_0}$, which establish an equivalence of categories.

Proof. This is an immediate consequence of Theorem 6 and Propositions 14 and 16. ■

7. STATES AS STRONGEST ACTUAL PROPERTIES

Let $S(\Sigma, Q, \eta)$ be a state-determined entity and let $(\Sigma, \mathcal{L}, \xi)$ be its state-determined state property system. Then it is possible to identify a state p of S with the strongest property it makes actual, i.e., with $\wedge\xi(p) \in \mathcal{L}$. As a consequence, one can embed $(\Sigma, <)$ into $(\mathcal{L}, <, \wedge, \vee)$ as an order-generating subset. This engenders another equivalence of categories.

We start by embedding $(\Sigma, <)$ into $(\mathcal{L}, <)$.

Theorem 8. Let $(\Sigma, \mathcal{L}, \xi)$ be a state property system. The following are equivalent:

(1) $(\Sigma, \mathcal{L}, \xi)$ is a state-determined state property system

(2) If we define

$$s_\xi: \quad \Sigma \to \mathcal{L}: \quad p \mapsto \wedge\xi(p) \tag{73}$$

then s_ξ is injective and for p, $q \in \Sigma$ we have

$$p < q \Leftrightarrow s_\xi(p) < s_\xi(q) \tag{74}$$

Therefore, if $(\Sigma, \mathcal{L}, \xi)$ is state determined, then s_ξ is isotone and injective and $(\Sigma, <)$ can be considered as a subposet of $(\mathcal{L}, <)$. We will use the notation $\Sigma^\xi = s_\xi(\Sigma)$.

Proof. ($2 \Rightarrow 1$) Equation (74) and the injectivity of s_ξ imply that $(\Sigma, <)$ is a poset, whence, through Proposition 14, ξ is injective.

($1 \Rightarrow 2$) We first verify (74). Suppose p, $q \in \Sigma$. Then $p < q \Leftrightarrow \xi(q) \subset \xi(p) \Leftrightarrow [s_\xi(q), I] \subset [s_\xi(p), I] \Leftrightarrow s_\xi(p) < s_\xi(q)$. Since the injectivity of ξ implies that $(\Sigma, <)$ is a poset, the injectivity of s_ξ follows from (74). ■

In the proof of Theorem 9 we will use the following result.

Proposition 17. Let $(\Sigma, \mathscr{L}, \xi)$ be a state property system. For a state $p \in \Sigma$ and a property $a \in \mathscr{L}$, we have, using the notation of Theorem 8, the following equivalence:

$$a \in \xi(p) \Leftrightarrow s_\xi(p) < a \tag{75}$$

Proof. ($\Rightarrow$) This implication follows immediately from the definition of s_ξ. ($\Leftarrow$) Suppose $s_\xi(p) < a$. Applying (17) for state p, we have that $s_\xi(p) = \wedge\xi\ (p) \in \xi(p)$ implies that $a \in \xi(p)$. ∎

Theorem 9. Let $(\Sigma, \mathscr{L}, \xi)$ be a state property system. Then $0 \notin \Sigma^\xi$ and Σ^ξ is an order-generating subset of $\mathscr{L}$: for every $a \in \mathscr{L}$ we have

$$a = \vee\{x \in \Sigma^\xi | x < a\} \tag{76}$$

Proof. Since $0 \notin \xi(p)$ for every p, we have $0 \notin \Sigma^\xi$. The '$>$' of equation (76) is trivial. To show '$<$', we will use equation (17). So, take $p \in \Sigma$ such that $a \in \xi(p)$. Then $s_\xi(p) < a$ and hence $s_\xi(p) \in \{x \in \Sigma^\xi | x < a\}$. This implies $s_\xi(p) < \vee\{x \in \Sigma^\xi | x < a\}$, or $\vee\{x \in \Sigma^\xi | x < a\} \in \xi(p)$. This proves $a < \vee\{x \in \Sigma^\xi | x < a\}$. ∎

We introduce some notation, which should make our intentions clear.

Definition 17. Let $(\Sigma, \mathscr{L}, \xi) \in$ **SP**. Then we put

$$H(\Sigma, \mathscr{L}, \xi) := (\Sigma^\xi, \mathscr{L}) \tag{77}$$

Now, let us try to go "back." First we introduce some terminology.

Definition 18. We call $(\Sigma, \mathscr{L})$ a 'based complete lattice' if $\mathscr{L}$ is a complete lattice and $\Sigma \subset \mathscr{L}$ is an order-generating subset not containing 0.

From Theorem 9 it follows that for every state property system $(\Sigma, \mathscr{L}, \xi)$, $H(\Sigma, \mathscr{L}, \xi)$ is a based complete lattice.

Theorem 10. Let $(\Sigma, \mathscr{L})$ be a based complete lattice. If we put for $p, q \in \Sigma$

$$p < q \Leftrightarrow p \prec q \ (\prec \text{is the order of } \mathscr{L}) \tag{78}$$

and

$$\xi: \quad \Sigma \to \mathscr{P}(\mathscr{L}) \tag{79}$$

$$p \mapsto \{a \in \mathscr{L} | p \prec a\} = [p, I] \tag{80}$$

then $(\Sigma, <, \mathscr{L}, \prec \wedge, \vee, \xi) =: K(\Sigma, \mathscr{L})$ is a state-determined state property system.

Proof. This proof mostly consists of very easy verifications. We will only make the following three remarks. For all $p \in \Sigma$, $0 \notin \xi(p)$ holds because

$0 \notin \Sigma$. The '$\Leftarrow$' of equation (17) is proven as follows: $a \in \xi(p) \Rightarrow b \in \xi(p)$ for every p in Σ implies that $\{p \in \Sigma \mid p \prec a\} \subset \{p \in \Sigma \mid p \prec b\}$. Since Σ is order-generating, this implies $a \prec b$. The state property system is state determined because $(\Sigma, <)$ is a poset. ■

To deal with the morphisms, we will use 'Galois connections'. We will quickly state the necessary results without proofs. Most of those proofs are straightforward. We will not give the results in their full generality, but will adapt them to the situation at hand. For more information we refer to Gierz *et al.* (1980).

Definition 19 (Galois connection). Let $\mathcal{L}$ and $\mathcal{L}'$ be complete lattices and let g: $\mathcal{L} \to \mathcal{L}'$ and d: $\mathcal{L}' \to \mathcal{L}$ be maps. (g, d) is a 'Galois connection' or an 'adjunction' between $\mathcal{L}$ and $\mathcal{L}'$ provided that

$$\forall (a, a') \in \mathcal{L} \times \mathcal{L}': \quad a' < g(a) \Leftrightarrow d(a') < a \tag{81}$$

g is called the 'upper adjoint' and d the 'lower adjoint' in (g, d). d is also called a lower adjoint of g and g an upper adjoint of d.

In fact, adjoints determine one another uniquely:

Theorem 11. Let $\mathcal{L}$ and $\mathcal{L}'$ be complete lattices and let n: $\mathcal{L} \to \mathcal{L}'$ and f: $\mathcal{L}' \to \mathcal{L}$ be maps. We have:

(1) n has a (necessarily unique) lower adjoint

$$n_*: \quad \mathcal{L}' \to \mathcal{L}: \quad a' \mapsto \wedge \{a \in \mathcal{L} | a' < n(a)\} \tag{82}$$

[i.e., n is an (the) upper adjoint of n_*] if and only if n preserves infima.

(2) f has a (necessarily unique) upper adjoint

$$f^*: \quad \mathcal{L} \to \mathcal{L}': \quad a \mapsto \vee \{a' \in \mathcal{L}' | f(a') < a\} \tag{83}$$

[i.e., f is a (the) lower adjoint of f^*] if and only if f preserves suprema.

This implies that if f preserves suprema, f^* exists and preserves infima, whence $(f^*)_*$ exists and equals f. Of course the "dual" holds for an infima-preserving n. ■

We remark that n: $\mathcal{L} \to \mathcal{L}'$ is said to 'preserve infima' if for every family $(a_i)_i \in \mathcal{L}$ we have $n(\wedge_i a_i) = \wedge_i n(a_i)$.

We introduce morphisms of based complete lattices and show their connection to morphisms of state property systems.

Definition 20 (morphism of based complete lattices). Let $(\Sigma, \mathcal{L})$ and $(\Sigma', \mathcal{L}')$ be based complete lattices. Then a function f: $\mathcal{L} \to \mathcal{L}'$ is called a 'morphism of based complete lattices' if

$$f(\Sigma) \subset \Sigma' \tag{84}$$

$$f(\vee_i a_i) = \vee_i f(a_i) \quad \forall (a_i)_i \in \mathcal{L} \tag{85}$$

The composition of these morphisms is given by the normal composition of functions.

Theorem 12. Consider $(m, n) \in \mathbf{SP}((\Sigma', \mathcal{L}', \xi'), (\Sigma, \mathcal{L}, \xi))$. Then

$$H(m, n) := n_*: \quad H(\Sigma', \mathcal{L}', \xi') \to H(\Sigma, \mathcal{L}, \xi) \tag{86}$$

is a morphism of based complete lattices.

Proof. Remember that $H(\Sigma, \mathcal{L}, \xi) = (\Sigma^{\xi}, \mathcal{L})$ (Definition 17). We know that n preserves infima [see (47)], whence it has a suprema-preserving lower adjoint n_*. Next, take $s_{\xi'}(p') \in \Sigma'^{\xi'}$. We have, for $a \in \mathcal{L}$, $s_{\xi'}(p') < n(a) \Leftrightarrow n(a) \in \xi'(p') \Leftrightarrow a \in \xi(m(p'))$. This implies $n_*(s_{\xi'}(p')) = \bigwedge\{a \in \mathcal{L} | s_{\xi'}(p') < n(a)\} = \bigwedge \xi(m(p')) = s_{\xi}(m(p')) \in \Sigma^{\xi}$. ■

Theorem 13. Let $f:(\Sigma', \mathcal{L}') \to (\Sigma, \mathcal{L})$ be a morphism of based complete lattices. Then

$$K(f): = (f|_{\Sigma'}^{\Sigma}, f^*): \quad K(\Sigma', \mathcal{L}') \to K(\Sigma, \mathcal{L}) \tag{87}$$

where $f|_{\Sigma'}^{\Sigma}: \Sigma' \to \Sigma$ is the restriction to Σ' and corestriction to Σ of f and $f^*: \mathcal{L} \to \mathcal{L}': a \mapsto \bigvee\{a' \in \mathcal{L}' | f(a') < a\}$, is a morphism of state property systems.

Proof. Remember that $K(\Sigma, \mathcal{L}) = (\Sigma, \mathcal{L}, \xi)$ with $\xi(p) = [p, I]$ (Theorem 10). Take $a \in \mathcal{L}$ and $p' \in \Sigma'$. Then $f^*(a) \in \xi'(p') \Leftrightarrow p' < f^*(a) \Leftrightarrow f(p') < a \Leftrightarrow a \in \xi(f(p')) = \xi(f|_{\Sigma'}^{\Sigma}(p))$. ■

We will need the following result.

Proposition 18. Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, be complete lattices and let $g_1: \mathcal{L}_1 \to \mathcal{L}_2$ and $g_2: \mathcal{L}_2 \to \mathcal{L}_3$ be two maps. If g_1 and g_2 are infima preserving, then so is $g_2 \circ g_1$ and

$$(g_2 \circ g_1)_* = (g_1)_* \circ (g_2)_* \tag{88}$$

Dually, if g_1 and g_2 are suprema preserving, then so is $g_2 \circ g_1$ and

$$(g_2 \circ g_1)^* = g_1^* \circ g_2^* \tag{89}$$

Proof. We only prove the first case. For $a \in \mathcal{L}_3$, $b \in \mathcal{L}_1$ we have $a < g_2 g_1(b) \Leftrightarrow (g_2)_*(a) < g_1(b) \Leftrightarrow (g_1)_*(g_2)_*(a) < b$. Using the uniqueness of adjoints, this proves our claim. ■

Since it is quite obvious that the composition of morphisms of based complete lattices yields again such a morphism, that it is associative, and that $id_{(\Sigma,\mathcal{L})}: = id_{\mathcal{L}}$ satisfies the necessary axioms, we can safely introduce the following category.

Definition 21 (category of based complete lattices). The category of based complete lattices with their morphisms is called $\mathbf{L_0}$.

We can now formally give the equivalence-establishing functors.

Theorem 14. The correspondence H: $\mathbf{SP} \to \mathbf{L_0}$ consisting of (1) the mapping

$$|\mathbf{SP}| \to |\mathbf{L_0}| \tag{90}$$

$$(\Sigma, \mathscr{L}, \xi) \mapsto H(\Sigma, \mathscr{L}, \xi) \tag{91}$$

where $H(\Sigma, \mathscr{L}, \xi)$ is the based complete lattice $(\Sigma^{\xi}, \mathscr{L})$ given by Theorem 8; and (2) for every pair of objects $(\Sigma, \mathscr{L}, \xi)$, $(\Sigma', \mathscr{L}', \xi')$ of $\mathbf{SP}$ the mapping

$$\mathbf{SP}((\Sigma', \mathscr{L}', \xi'),(\Sigma, \mathscr{L}, \xi)) \to \mathbf{L_0}(H(\Sigma', \mathscr{L}', \xi'), H(\Sigma, \mathscr{L}, \xi)) \tag{92}$$

$$(m, n) \mapsto H(m, n) = n_* \tag{93}$$

is a covariant functor.

Proof. This is, apart from some minor checks, a consequence of Theorems 9 and 12 and Proposition 18. ■

Theorem 15. The correspondence K: $\mathbf{L_0} \to \mathbf{SP}$ consisting of (1) the mapping

$$|\mathbf{L_0}| \to |\mathbf{SP}| \tag{94}$$

$$(\Sigma, \mathscr{L}) \mapsto K(\Sigma, \mathscr{L}) = (\Sigma, \mathscr{L}, \xi) \tag{95}$$

where $\xi(p) = [p,I]$ (Proposition 10); and (2) for every pair of objects $(\Sigma, \mathscr{L})$, $(\Sigma', \mathscr{L}')$ of $\mathbf{L_0}$ the mapping

$$\mathbf{L_0}((\Sigma', \mathscr{L}'), (\Sigma, \mathscr{L})) \to \mathbf{SP}(K(\Sigma', \mathscr{L}'), K(\Sigma, \mathscr{L})) \tag{96}$$

$$f \mapsto K(f) = (f|_{\Sigma'}^{\Sigma}, f^*) \tag{97}$$

is a covariant functor.

Proof. This is, apart from some minor checks, a consequence of Theorems 10 and 13 and Proposition 18. ■

Finally, we reach the following result.

Theorem 16 (equivalence of $\mathbf{SP_0}$ and $\mathbf{L_0}$). The covariant functor H restricts to the functor H: $\mathbf{SP_0} \to \mathbf{L_0}$ and the covariant functor K corestricts to the functor K: $\mathbf{L_0} \to \mathbf{SP_0}$. These functors establish an equivalence of categories. Moreover, $H \circ K = Id_{\mathbf{L_0}}$.

Proof. First we remark that K corestricts to the functor K: $\mathbf{L_0} \to \mathbf{SP_0}$ by Proposition 10.

Step 1: $K \circ H$. Consider $(\Sigma, \mathscr{L}, \xi) \in |\mathbf{SP_0}|$. Then $KH(\Sigma, \mathscr{L}, \xi) = (\Sigma^\xi, \mathscr{L}, \bar{\xi})$ with

$$\bar{\xi}: \Sigma^\xi \to \mathscr{P}(\mathscr{L}) \tag{98}$$

$$a_p = s_\xi(p) \mapsto [a_p, I] \tag{99}$$

Also, if $(m, n) \in \mathbf{SP_0}((\Sigma', \mathscr{L}', \xi'), (\Sigma, \mathscr{L}, \xi))$, then

$$KH(m, n) = K(n_*) = (n_*|^{\Sigma^\xi}_{\Sigma'^{\xi'}}, n)$$

Step 2: $Id_{\mathbf{SP_0}} \cong KH$. For $(\Sigma, \mathscr{L}, \xi) \in |\mathbf{SP_0}|$ define

$$\eta_{(\Sigma,\mathscr{L},\xi)}: \quad (\Sigma, \mathscr{L}, \xi) \to KH(\Sigma, \mathscr{L}, \xi) \tag{100}$$

$$\eta_{(\Sigma,\mathscr{L},\xi)} = (s_\xi, id_{\mathscr{L}}) \tag{101}$$

Then $\eta = (\eta_{(\Sigma,\mathscr{L},\xi)}): Id_{\mathbf{SP_0}} \to KH$ is a natural isomorphism. First we verify that $\eta_{(\Sigma,\mathscr{L},\xi)}$ is a morphism of $\mathbf{SP_0}$. Indeed, for $a \in \mathscr{L}$ and $p \in \Sigma$ we have $a \in \bar{\xi}(s_\xi(p)) \Leftrightarrow s_\xi(p) < a \Leftrightarrow id_{\mathscr{L}}(a) = a \in \xi(p)$. To show that $(s_\xi, id_{\mathscr{L}})$ is an isomorphism, we only have to prove that $s_\xi: \Sigma \to \Sigma^\xi$ is bijective (Proposition 13) and this follows from $\Sigma^\xi = s_\xi(\Sigma)$ and Proposition 8. Finally we prove the naturality of η. Take $(m, n) \in \mathbf{SP_0}((\Sigma', \mathscr{L}', \xi'), (\Sigma, \mathscr{L}, \xi))$. We have to show $n_* \circ s_{\xi'} = s_\xi \circ m$. This has been done in the proof of Theorem 12.

Step 3: $HK = Id_{\mathbf{L_0}}$. Let f be a morphism of $\mathbf{L_0}$. We have

$$f \overset{K}{\mapsto} (f|^{\Sigma}_{\Sigma'}, f^*) \overset{H}{\mapsto} (f^*)_* = f \tag{102}$$

Next, consider a based complete lattice $(\Sigma, \mathscr{L})$. Then $K(\Sigma, \mathscr{L}) = (\Sigma, \mathscr{L}, \xi)$ with $\xi(p) = [p, I]$ for $p \in \Sigma$. This implies that $s_\xi(p) = \wedge\, \xi(p) = p$, whence $\Sigma^\xi = \Sigma$. Therefore $HK(\Sigma, \mathscr{L}) = H(\Sigma, \mathscr{L}, \xi) = (\Sigma^\xi, \mathscr{L}) = (\Sigma, \mathscr{L})$. ■

Theorem 17. We have the following equivalences of categories:

$$\mathbf{Cls} \approx \mathbf{SP} \tag{103}$$

$$\mathbf{Cls_0} \approx \mathbf{SP_0} \approx \mathbf{L_0} \tag{104}$$

This last theorem shows that a state-determined entity can "equivalently" be described by a T_0 closure space (where the states are the points—or the point closures—and the properties are represented by the closed subsets), a state-determined state property system, or a based complete lattice (where the states form an order-generating subset of the property lattice).

Erné (1984) shows the direct equivalence between $\mathbf{Cls_0}$ and $\mathbf{L_0}$.

8. CONSTRUCTION OF THE (CO)PRODUCT OF TWO STATE PROPERTY SYSTEMS

We will now construct the product of two state property systems in **SP**. For the necessary category theory we refer to Borceux (1994).

Theorem 18. Let $(\Sigma_1, \mathcal{L}_1, \xi_1)$ and $(\Sigma_2, \mathcal{L}_2, \xi_2)$ be state property systems (objects of **SP**). Then $(P, (s_1, s_2))$ is the [up to isomorphism (see Borceux (1994), Proposition 2.2.2)] product of $(\Sigma_1, \mathcal{L}_1, \xi_1)$ and $(\Sigma_2, \mathcal{L}_2, \xi_2)$ in **SP**, where P is the state property system $(\Sigma, <, \mathcal{L}, \xi)$ with

$$\Sigma = \Sigma_1 \times \Sigma_2 \tag{105}$$

$$(p_1, p_2) < (q_1, q_2) \Leftrightarrow p_1 < q_1 \text{ and } p_2 < q_2 \text{ for } p_i, q_i \in \Sigma_i \tag{106}$$

$$\mathcal{L} = \mathcal{L}_1 \sqcup \mathcal{L}_2 \tag{107}$$

$$= \{(a_1, a_2) | a_1 \in \mathcal{L}_1, a_2 \in \mathcal{L}_2, a_1 \neq 0_1, a_2 \neq 0_2\} \cup \{0\} \tag{108}$$

equipped with the following partial order relation:

$$(a_1, a_2) < (b_1, b_2) \Leftrightarrow a_1 < b_1 \text{ and } a_2 < b_2 \tag{109}$$

$$0 < (a_1, a_2) \qquad \text{for all} \quad (a_1, a_2) \tag{110}$$

and lattice operations

$$\bigwedge_i (a_1^i, a_2^i) = \begin{cases} (\wedge_i a_1^i, \wedge_i a_2^i) & \text{if} \quad \wedge_i a_1^i \neq 0_1 \text{ and } \wedge_i a_2^i \neq 0_2 \\ 0 & \text{otherwise} \end{cases} \tag{111}$$

$$\bigvee_i (a_1^i, a_2^i) = (\vee_i a_1^i, \vee_i a_2^i) \tag{112}$$

and with

$$\xi: \quad \Sigma \to \mathcal{P}(\mathcal{L}) \tag{113}$$

$$(p_1, p_2) \mapsto \{(a_1, a_2) \in \mathcal{L} | a_1 \in \xi_1 (p_1), a_2 \in \xi_2(p_2)\} \tag{114}$$

and $s_i = (\pi_i, \iota_i)$ with

$$\pi_i: \quad \Sigma \to \Sigma_i: \quad (p_1, p_2) \mapsto p_i \tag{115}$$

$$\iota_1: \quad \mathcal{L}_1 \to \mathcal{L}_1 \sqcup \mathcal{L}_2 \tag{116}$$

$$a_1 \mapsto (a_1, I_2) \qquad \text{if} \quad a_1 \neq 0_1 \tag{117}$$

$$0_1 \mapsto 0 \tag{118}$$

$$\iota_2: \quad \mathcal{L}_2 \to \mathcal{L}_1 \sqcup \mathcal{L}_2 \tag{119}$$

$$a_2 \mapsto (I_1, a_2) \qquad \text{if} \quad a_2 \neq 0_2 \tag{120}$$

$$0_2 \mapsto 0 \tag{121}$$

Proof. Step 1: $P \in |\mathbf{SP}|$. We have to check the conditions of Definition 9. After noting that $\xi(p_1, p_2) = \xi_1(p_1) \times \xi_2(p_2)$, $I = (I_1, I_2)$ this requires more writing than thinking.

Step 2: s_i is a morphism of **SP**. We check equation (43). Let $(p_1, p_2) \in \Sigma$, $a_1 \in \mathcal{L}_1$, and take $a_1 \neq 0_1$ (the other case is trivial). Then $\iota_1(a_1) = (a_1, I_2) \in \xi(p_1, p_2) \Leftrightarrow a_1 \in \xi_1(p_1) = \xi_1(\pi_1(p_1, p_2))$ [$I_2 \in \xi_2(p_2)$ always holds].

Step 3: Let $Q = (\Sigma', \mathcal{L}', \xi')$ be a state property system and consider two morphisms of **SP**: (m_1, n_1): $Q \to (\Sigma_1, \mathcal{L}_1, \xi_1)$ and (m_2, n_2): $Q \to (\Sigma_2, \mathcal{L}_2, \xi_2)$. We define (m, n) by

$$m: \quad \Sigma' \to \Sigma: \quad p' \mapsto (m_1(p'), m_2(p')) \tag{122}$$

$$n: \quad \mathcal{L} \to \mathcal{L}': \quad (a_1, a_2) \mapsto n_1(a_1) \wedge n_2(a_2) \tag{123}$$

$$0 \mapsto 0' \tag{124}$$

Then (m, n): $Q \to P$ is a morphism of **SP**. Indeed for $a_i \in \mathcal{L}_i$, $a_i \neq 0_i$, $i = 1, 2$ (the zero case is trivial) and $p' \in \Sigma'$ we have $(a_1, a_2) \in \xi(m(p')) = \xi(m_1(p'), m_2(p')) \Leftrightarrow a_1 \in \xi_1(m_1(p'))$, $a_2 \in \xi_2(m_2(p')) \Leftrightarrow n_1(a_1) \in \xi'(p')$, $n_2(a_2) \in \xi'(p') \Leftrightarrow n(a_1, a_2) = n_1(a_1) \wedge n_2(a_2) \in \xi'(p')$.

Step 4: $s_i \circ (m, n) = (m_i, n_i)$. We have to show $\pi_i \circ m = m_i$ and $n \circ \iota_i = n_i$. The first is trivial. The second is not difficult either: for $a_1 \neq 0_1$ (other case again trivial) we have $n(\iota_1(a_1)) = n(a_1, I_2) = n_1(a_1) \wedge n_2(I_2) = n_1(a_1)$ since $n_2(I_2) = I'$.

Step 5: We have to show that (m, n) is the only morphism such that $(m_i, n_i) = s_i \circ (m, n)$. Clearly m is the only function such that $m_i = \pi_i \circ m$. Now, $n_i = n \circ \iota_i$ clearly implies that for $a_i \in \mathcal{L}_i$, $a_i \neq 0$ [$n(0) = 0'$ must hold because n should be a morphism] we have $n(a_1, a_2) = n((a_1, I_2) \wedge (I_1, a_2)) = n(a_1, I_2) \wedge n(I_1, a_2) = n(\iota_1(a_1)) \wedge n(\iota_2(a_2)) = n_1(a_1) \wedge n_2(a_2)$. ■

We make some remarks. (1) If we consider the opposite category $\mathbf{SP}^{\mathrm{op}}$, this product becomes a coproduct. This is a generalization of the coproduct (tensor product) of property lattices of Aerts (1984a), which is in fact a product in $\mathbf{L_0}$ (or a coproduct in $\mathbf{L_0^{op}}$). (2) As the finite coproduct of Aerts (1984a) has been generalized to arbitrary coproducts (Aerts and Valckenborgh, 1998), the product of the previous theorem can also be constructed for arbitrary families of state property systems. (3) Even before we did the calculations for the previous theorem, we knew the category **SP** had arbitrary products, since it is equivalent to the topological (and hence complete) category **Cls**.

REFERENCES

Aerts, D. (1981). The one and the many, Doctoral Thesis, Brussels Free University.

Aerts, D. (1982). Description of many physical entities without the paradoxes encountered in quantum mechanics, *Found. Phys.* **12**, 1131.

Aerts, D. (1983a). Classical theories and non classical theories as a special case of a more general theory, *J. Math. Phys.* **24**, 2441.
Aerts, D. (1983b). The description of one and many physical systems, in *Foundations of Quantum Mechanics*, C. Gruber, ed., A.V.C.P., Lausanne, p. 63.
Aerts, D. (1984a). Construction of the tensor product for the lattices of properties of physical entities, *J. Math. Phys.* **25**, 1434.
Aerts, D. (1984b). The missing elements of reality in the description of quantum mechanics of the EPR paradox situation, *Helv. Phys. Acta* **57**, 421.
Aerts, D. (1985). The physical origin of the EPR paradox and how to violate Bell inequalities by macroscopic systems, in *Symposium on the Foundations of Modern Physics*, P. Mittelstaedt, and P. Lahti, eds., World Scientific, Singapore.
Aerts, D. (1994). Quantum structures, separated physical entities and probability, *Found. Phys.* **24**, 1227.
Aerts, D. (1998). Foundations of quantum physics: A general realistic and operational approach, *Int. J. Theor. Phys.*, this issue.
Aerts, D., and Valckenborgh, F. (1998). Lattice extensions and the description of compound entities, FUND, Brussels Free University, Preprint.
Aerts, D., Coecke, B., Durt, T., and Valckenborgh, F. (1997a). Quantum, classical and intermediate; a model on the Poincaré sphere, *Tatra Mountains Math. Publ.* **10**, 225.
Aerts, D., Coecke, B., Durt, T., and Valckenborgh, F. (1997b). Quantum, classical and intermediate; the vanishing vector space structure, *Tatra Mountains Math. Publ.* **10**, 241.
Borceux F. (1994). *Handbook of Categorical Algebra I*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge.
Cattaneo, G., and Nistico, G. (1991). Axiomatic foundations of quantum physics: Critiques and misunderstandings. Piron's question-proposition system, *Int. J. Theor. Phys.* **30**, 1293.
Cattaneo, G., and Nistico, G. (1992). Physical content of preparation-question structures and Brouwer–Zadeh lattices, *Int. J. Theor. Phys.* **31**, 1873.
Cattaneo, G., and Nistico, G. (1993). A model of Piron's preparation-question structures in Ludwig's selection structures, *Int. J. Theor. Phys.* **32**, 407.
Cattaneo G., dalla Pozza C., Garola C., and Nistico G. (1988). On the logical foundations of the Jauch–Piron approach to quantum physics, *Int. J. Theor. Phys.* **27**, 1313.
Daniel, W. (1982). On non-unitary evolution of quantum systems, *Helv. Phys. Acta* **55**, 330.
d'Emma, G. (1980). On quantization of the electromagnetic field, *Helv. Phys. Acta* **53**, 535.
Erné M. (1984). Lattice representations for categories of closure spaces, in *Categorical Topology*, Heldermann Verlag, Berlin, pp. 197–122.
Foulis, D., and Randall, C. (1984). A note on the misunderstanding of Piron's axioms for quantum mechanics, *Found. Phys.* **14**, 65.
Foulis, D., Piron, C., and Randall, C. (1983). Realism, operationalism and quantum mechanics, *Found. Phys.* **13**, 843.
Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M., and Scott, D. S. (1980). *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin.
Giovannini, N., and Piron, C. (1979). On the group-theoretical foundations of classical and quantum physics: Kinematics and state spaces, *Helv. Phys. Acta* **52**, 518.
Gisin, N. (1981). Spin relaxation and dissipative Schrödinger like evolution equations, *Helv. Phys. Acta* **54**, 457.
Jauch, J. M., and Piron, C. (1965). Generalized localisability, *Helv. Phys. Acta* **38**, 104.
Ludwig, G., and Neumann, H. (1981). Connections between different approaches to the foundations of quantum mechanics, in *Interpretation and Foundations of Quantum Mechanics*, H. Neumann, ed., B. I. Wissenschafts-Verlag, Bibliographisches Institut, Mannheim.

Moore, D. (1995). "Categories of Representations of Physical Systems", *Helv. Phys. Acta,* **68**, 658.

Piron, C. (1964). Axiomatique quantique, *Helv. Phys. Acta* **37**, 439.

Piron, C. (1969). Les régles de superselection continues, *Helv. Phys. Acta* **43**, 330.

Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts.

Piron, C. (1989). Recent developments in quantum mechanics, *Helv. Phys. Acta* **62**, 82.

Piron, C. (1990). *Mécanique Quantique: Bases et Applications*, Presses polytechniques et universitaires romandes, Lausanne, Switzerland.

Randall, C., and Foulis, D. (1983). Properties and operational propositions in quantum mechanics, *Found. Phys.* **13**, 843.